# BU CS 332 – Theory of Computation

## Lecture 24:

- Space Complexity

Reading:

Sipser Ch 8.1-8.3

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# Space analysis

Space complexity of a TM (algorithm) = maximum number of tape cell it uses on a worst-case input

Formally: Let $f : \mathbb{N} \rightarrow \mathbb{N}$. A TM $M$ runs in space $f(n)$ if on every input $w \in \Sigma^*$, $M$ halts on $w$ using at most $f(n)$ cells

For nondeterministic machines: Let $f : \mathbb{N} \rightarrow \mathbb{N}$. An NTM $N$ runs in space $f(n)$ if on every input $w \in \Sigma^*$, $N$ halts on $w$ using at most $f(n)$ cells on every computational branch

# Space complexity classes

Let $f : \mathbb{N} \to \mathbb{N}$

A language $A \in \mathrm{SPACE}(f(n))$ if there exists a basic single-tape (deterministic) TM $M$ that

1) Decides $A$, and
2) Runs in space $O(f(n))$

A language $A \in \mathrm{NSPACE}(f(n))$ if there exists a single-tape nondeterministic TM $N$ that

1) Decides $A$, and
2) Runs in space $O(f(n))$

# Space vs. Time

*We saw:*

$$TIME\big(f(n)\big) \subseteq NTIME\big(f(n)\big) \subseteq SPACE\big(f(n)\big) \subseteq TIME(2^{O\left(f(n)\right)})$$

# What is the space-cost of non-determinism?

For time, the best we can do is:

$$TIME\big(f(n)\big) \subseteq NTIME\big(f(n)\big) \subseteq TIME(2^{O(f(\mathrm{n}))})$$

Can we do better for space?

# Savitch's Theorem: Deterministic vs. Nondeterministic Space

Theorem: Let $f$ be a function with $f(n) \geq n$. Then $NSPACE\big(f(n)\big) \subseteq SPACE\left(\big(f(n)\big)^2\right)$.

Proof idea:

- Let $N$ be an NTM deciding $f$ in space $f(n)$
- We construct a TM $M$ deciding $f$ in space $O\left(\big(f(n)\big)^2\right)$
- Actually solve a more general problem:

We will design procedure CANYIELD$(c_1, c_2, t)$ :

- Given configurations $c_1, c_2$ of $N$ and natural number $t$, decide whether $N$ can go from $c_1$ to $c_2$ in $\leq t$ steps on some nondeterministic path.

# Savitch's Theorem

- Let $N$ be an NTM deciding $f$ in space $f(n)$

$M$ = "On input $w$:

Output the result of CANYIELD$(c_s, c_a, 2^{df(n)})$"

# Savitch's Theorem

CANYIELD$(c_1, c_2, t)$ decides whether $N$ can go from configuration $c_1$ to $c_2$ in $\leq t$ steps on some nondeterministic path:

CANYIELD$(c_1, c_2, t)$ =

1. If $t = 1$, accept if $c_1 = c_2$ or $c_1$ yields $c_2$ in one transition.
   Else, reject.
2. If $t > 1$, then for each config $c_{mid}$ of $N$ with $\leq f(n)$ cells:
3. Run CANYIELD($\langle c_1, c_{mid}, t/2\rangle$).
4. Run CANYIELD($\langle c_{mid}, c_2, t/2\rangle$).
5. If both runs accept, accept.
6. Reject.

# Complexity class PSPACE

**Definition:** PSPACE is the class of languages decidable in polynomial space on a basic single-tape (deterministic) TM

$$\mathrm{PSPACE} = \bigcup_{k=1}^{\infty} \mathrm{SPACE}(n^k)$$

**Definition:** NPSPACE is the class of languages decidable in polynomial space on a single-tape (nondeterministic) TM

$$\mathrm{NPSPACE} = \bigcup_{k=1}^{\infty} \mathrm{NSPACE}(n^k)$$

# Relationships between complexity classes

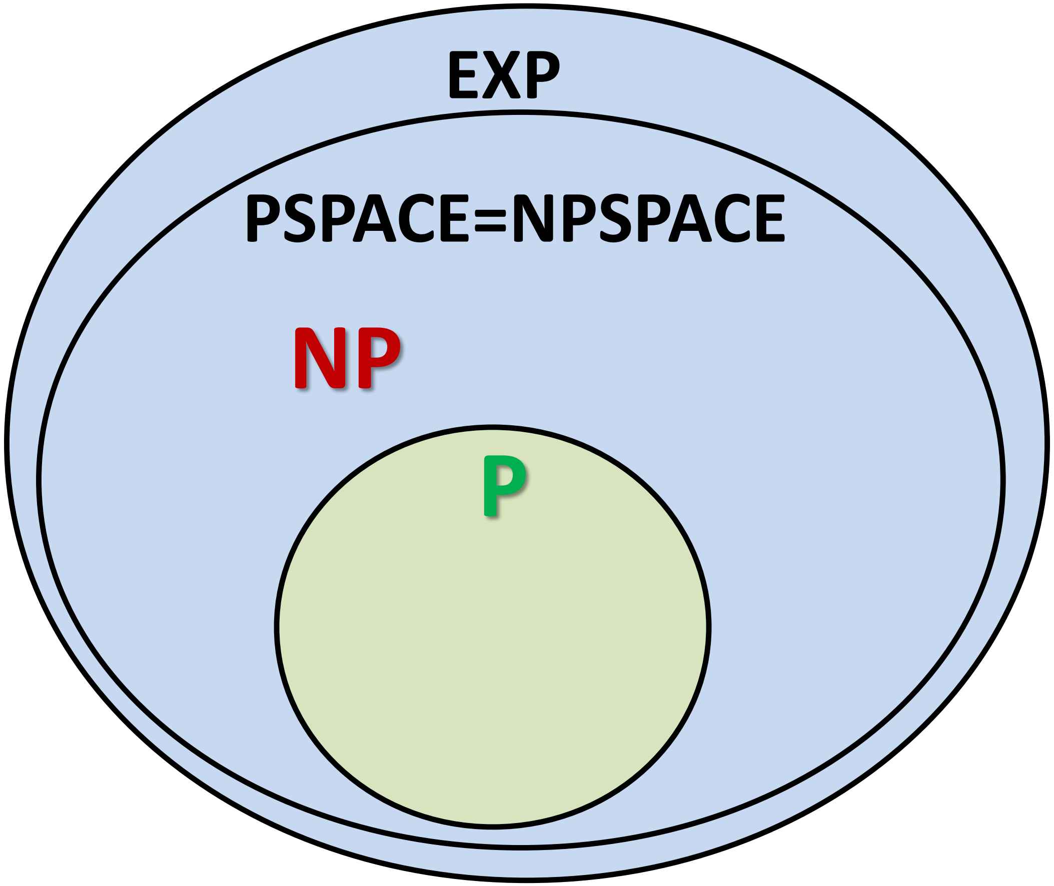
1. $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXP}$

since $SPACE\big(f(n)\big) \subseteq TIME(2^{O(f(n))})$

2. $\mathrm{P} \neq \mathrm{EXP}$ (Monday)

Which containments in (1) are proper?

**Unknown!**



# PSPACE-Completeness

# What happens in a world where $\mathrm{P} \neq \mathrm{PSPACE}$?

Even more believable than $\mathrm{P} \neq \mathrm{NP}$, but still(!) very far from proving it

Question: What would $\mathrm{P} \neq \mathrm{PSPACE}$ allow us to conclude about problems we care about?

PSPACE-completeness: Find the "hardest" problems in PSPACE

Find $L \in \mathrm{PSPACE}$ such that $L \in \mathrm{P}$ iff $\mathrm{P} = \mathrm{PSPACE}$

# Reminder: NP-completeness

Definition: A language $B$ is NP-complete if

1) $B \in \mathrm{NP}$, and
2) Every language $A \in \mathrm{NP}$ is poly-time reducible to $B$, i.e., $A \leq_{\mathrm{p}} B$ (“$B$ is NP-hard”)

# PSPACE-completeness

Definition: A language $B$ is PSPACE-complete if

1) $B \in \text{PSPACE}$, and
2) Every language $A \in \text{PSPACE}$ is poly-time reducible to $B$, i.e., $A \leq_{\mathrm{p}} B$ ("$B$ is PSPACE-hard")

# A PSPACE-complete problem: TQBF

"Is a fully quantified logical formula true?"

- Boolean variable: Variable that can take on the value true/false (encoded as 0/1)
- Boolean operations: $\wedge$ (AND), $\vee$ (OR), $\neg$ (NOT)
- Boolean formula: Expression made of Boolean variables and operations. Ex: $(x_1 \vee \overline{x_2}) \wedge x_3$
- Fully quantified Boolean formula: Boolean formula with all variables quantified ($\forall, \exists$) Ex: $\forall x_1 \exists x_3 \forall x_2 \quad (x_1 \vee \overline{x_2}) \wedge x_3$
- Every fully quantified Boolean formula is either true or false

- $TQBF = \{\langle\varphi\rangle | \varphi \text{ is a true fully quantified formula}\}$

# Theorem: TQBF is PSPACE-complete

Need to prove two things…

1) $TQBF \in \mathrm{PSPACE}$

2) Every problem in $\mathrm{PSPACE}$ is poly-time reducible to $TQBF$ ($TQBF$ is PSPACE-hard)

# 1) TQBF is in PSPACE

$T$ = "On input $\langle \varphi \rangle$,
where $\varphi$ is a fully quantified Boolean formula:

1. If $\varphi$ has no quantifiers, it has only constants (and no variables). Evaluate $\varphi$.
   If true, accept; else, reject.
2. If $\varphi$ is of the form $\exists x\ \psi$, recursively call $T$ on $\psi$ with $x = 0$ and then on $\psi$ with $x = 1$.
   If either call accepts, accept; else, reject.
3. If $\varphi$ is of the form $\forall x\ \psi$, recursively call $T$ on $\psi$ with $x = 0$ and then on $\psi$ with $x = 1$.
   If both calls accept, accept; else, reject."

- If $n$ is the input length, $T$ uses space $O(n)$.

# 2) TQBF is PSPACE-hard

Theorem: Every language $A \in \mathrm{PSPACE}$ is poly-time reducible to $TQBF$

Proof idea:

Let $A \in \mathrm{PSPACE}$ be decided by a poly-space deterministic TM $M$. Using proof of Cook-Levin Theorem,

$$M \text{ accepts input } w \Leftrightarrow \text{ formula } \varphi_{M,w} \text{ is true}$$

Using idea of Savitch's Theorem, replace $\varphi_{M,w}$ with a quantified formula of poly-size that can be computed in poly-time

# Unconditional Hardness

# Hardness results so far

- If $\mathrm{P} \neq \mathrm{NP}$, then $3SAT \notin P$
- If $\mathrm{P} \neq \mathrm{PSPACE}$, then $TQBF \notin P$

Question: Are there decidable languages that we can show are not in $P$?

# Diagonalization redux

| TM $M$ | | | | | | |
|---|---|---|---|---|---|---|
| $M_1$ | | | | | | |
| $M_2$ | | | | | | |
| $M_3$ | | | | | | |
| $M_4$ | | | | | | |
| $\vdots$ | | | | | | |
| | | | | | | |

# Diagonalization redux

| TM $M$ | $M(\langle M_1 \rangle)$? | $M(\langle M_2 \rangle)$? | $M(\langle M_3 \rangle)$? | $M(\langle M_4 \rangle)$? | | $D(\langle D \rangle)$? |
|---|---|---|---|---|---|---|
| $M_1$ | Y | N | Y | Y | … | |
| $M_2$ | N | N | Y | Y | | |
| $M_3$ | Y | Y | Y | N | | |
| $M_4$ | N | N | Y | N | | |
| $\vdots$ | | | | | $\ddots$ | |
| $D$ | | | | | | |

$\overline{SA_{\text{TM}}} = \{\langle M \rangle \mid M$ is a TM that does not accept input $\langle M \rangle\}$

$\overline{SA_{\text{TM},EXP}} = \{\langle M \rangle \mid M$ is a TM that does not accept input $\langle M \rangle$ within $2^{|\langle M \rangle|}$ steps$\}$

# An explicit undecidable language

- Theorem: $L = \overline{SA_{\mathrm{TM},EXP}} = \{\langle M\rangle \mid M \text{ is a TM that does not accept input } \langle M\rangle \text{ within } 2^{|\langle M\rangle|} \text{ steps}\}$

is in $\mathrm{EXP}$, but not in $\mathrm{P}$

Proof:

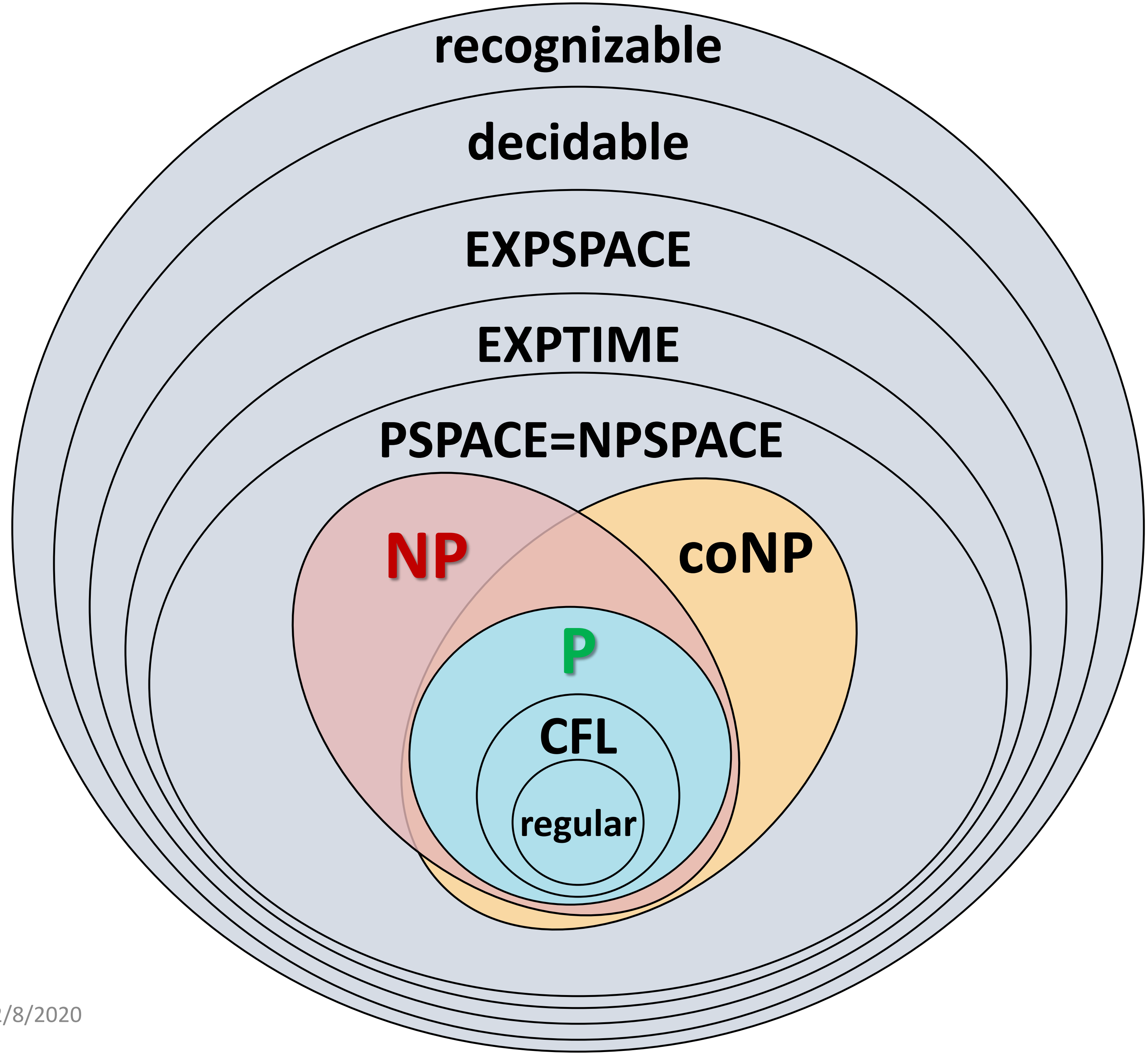
- In $\mathrm{EXP}$: Simulate $M$ on input $\langle M\rangle$ for $2^{|\langle M\rangle|}$ steps and flip its decision
- Not in $\mathrm{P}$: Suppose for contradiction that $D$ decides $L$ in time $n^k$

# Time and space hierarchy theorems

- For any* function $f(n) \geq n \log n$, a language exists that is decidable in $f(n)$ time, but not in $o\left(\frac{f(n)}{\log f(n)}\right)$ time.

- For any* function $f(n) \geq n \log n$, a language exists that is decidable in $f(n)$ space, but not in $o(f(n))$ space.

*time constructible and space constructible, respectively


recognizable
decidable
EXPSPACE
EXPTIME
PSPACE=NPSPACE
NP
coNP
P
CFL
regular