BU CS 332 – Theory of Computation

Lecture 14:

- Unrecognizability
- Undecidability

Reading:

Sipser Ch. 4

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October 27, 2020

A general theorem about set sizes

Theorem: Let X be a set. Then the power set $P(X)$ does **not** have the same size as X.

Proof: Assume for the sake of contradiction that there is a correspondence $f: X \to P(X)$

Construct a set $S \in P(X)$ that cannot be the output $f(x)$ for any $x \in X$:

$$S = \{x \in X \mid x \notin f(x)\}$$

If $S = f(y)$ for some $y \in X$,

then $y \in S$ if and only if $y \notin S$

Diagonalization argument

Assume a correspondence $f: X \to P(X)$

x	$x_1 \in f(x)$?	$x_2 \in f(x)$?	$x_3 \in f(x)$?	$x_4 \in f(x)$?	$\dots$
x_1	Y	N	Y	Y	
x_2	N	N	Y	Y	
x_3	Y	Y	Y	N	
x_4	N	N	Y	N	
$\vdots$					$\ddots$

Define S by flipping the diagonal:

Put $\quad x_i \in S \quad \Longleftrightarrow \quad x_i \notin f(x_i)$

An Existential Proof

Theorem: There exists an unrecognizable language over $\{0, 1\}$

Proof:

Set of all Turing machines: $X \subseteq \{0, 1\}^*$

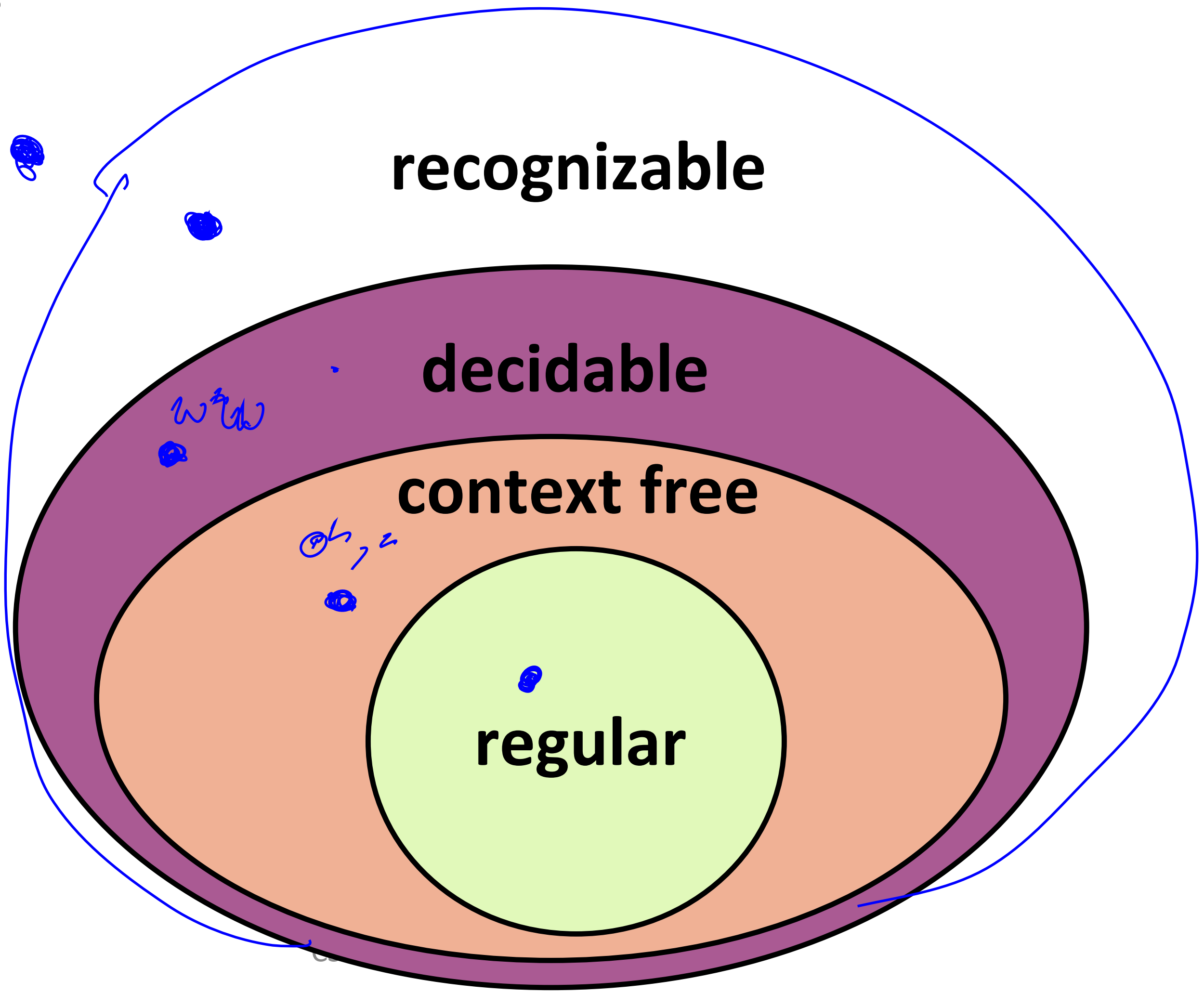
Set of all languages over $\{0, 1\}$ = all subsets of $\{0, 1\}^*$

= $P(X)$

There are more languages than there are TMs!

Questions

- Are there languages that are recognizable but not decidable?



Questions

- Are there languages that are recognizable but not decidable?

- Are there any languages of interest that are unrecognizable/undecidable?

A Specific Undecidable Language

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Theorem: A_{TM} is undecidable

Proof: Assume for the sake of contradiction that TM H decides A_{TM}:

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

Diagonalization: Use H to check what M does when given as input its own description…and do the opposite

A Specific Undecidable Language

$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Suppose H decides A_{TM}

$$D(\langle M\rangle) = \begin{cases} Y & \text{if } M(\langle M\rangle) = N \\ N & \text{if } M(\langle M\rangle) = Y \end{cases}$$

Consider the following TM D.

On input $\langle M\rangle$ where M is a TM:

1. Run H on input $\langle M, \langle M\rangle\rangle$
2. If H accepts, reject. If H rejects, accept.

Question: What does D do on input $\boxed{\langle D\rangle}$?

D never halts

How is this diagonalization?

TM M					
M_1					
M_2					
M_3					
M_4					
$\vdots$					

How is this diagonalization?

TM M	$M(\langle M_1 \rangle)$?	$M(\langle M_2 \rangle)$?	$M(\langle M_3 \rangle)$?	$M(\langle M_4 \rangle)$?	…
M_1	Y	N	Y	Y	
M_2	N	N	Y	Y	
M_3	Y	Y	Y	N	
M_4	N	N	Y	N	
$\vdots$					$\ddots$

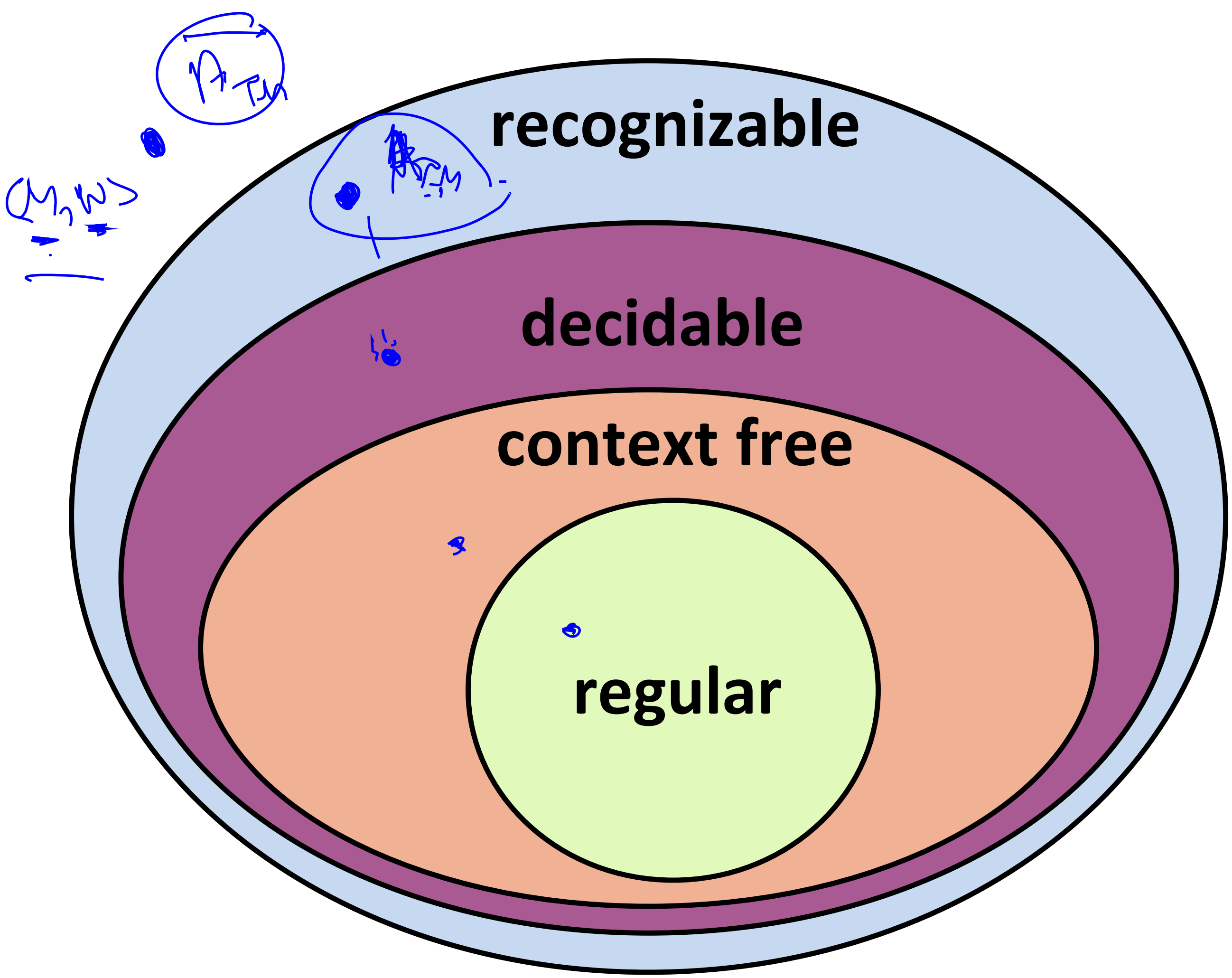
D accepts input $\langle M_i \rangle$ $\Leftrightarrow$ M_i does not accept input $\langle M_i \rangle$

How is this diagonalization?

TM M	$M(\langle M_1 \rangle)$?	$M(\langle M_2 \rangle)$?	$M(\langle M_3 \rangle)$?	$M(\langle M_4 \rangle)$?		$D(\langle D \rangle)$?
M_1	Y	N	Y	Y	…	
M_2	N	N	Y	Y		
M_3	Y	Y	Y	N		
M_4	N	N	Y	N		
$\vdots$					$\ddots$	
D						

D accepts input $\langle M_i \rangle$ $\Leftrightarrow$ M_i does not accept input $\langle M_i \rangle$

Classes of Languages: updated view



Recall!!

L is decidable iff L & $\bar{L}$ are recognizable.

A specific unrecognizable Language

Theorem: A language L is decidable if and only if L and $\overline{L}$ are both Turing-recognizable.

Proof:

Assume L, $\overline{L}$ are recognizable

if $x \in L \Rightarrow M$

$x \notin \overline{L} \Rightarrow$

steps

A specific unrecognizable Language

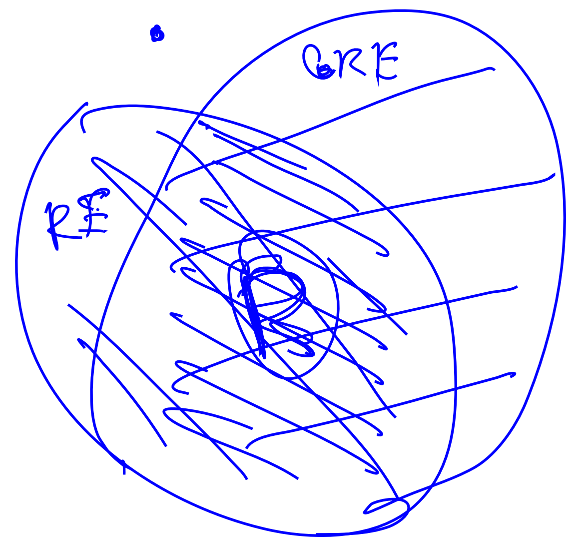
Theorem: A language L is decidable if and only if L and $\overline{L}$ are both Turing-recognizable.

Corollary: If L is Turing-recognizable and undecidable then $\overline{L}$ is not Turing-recognizable.

A specific unrecognizable Language

Theorem: A language L is decidable if and only if L and $\overline{L}$ are both Turing-recognizable.

Corollary: If L is Turing-recognizable and undecidable then $\overline{L}$ is not Turing-recognizable.



Define:

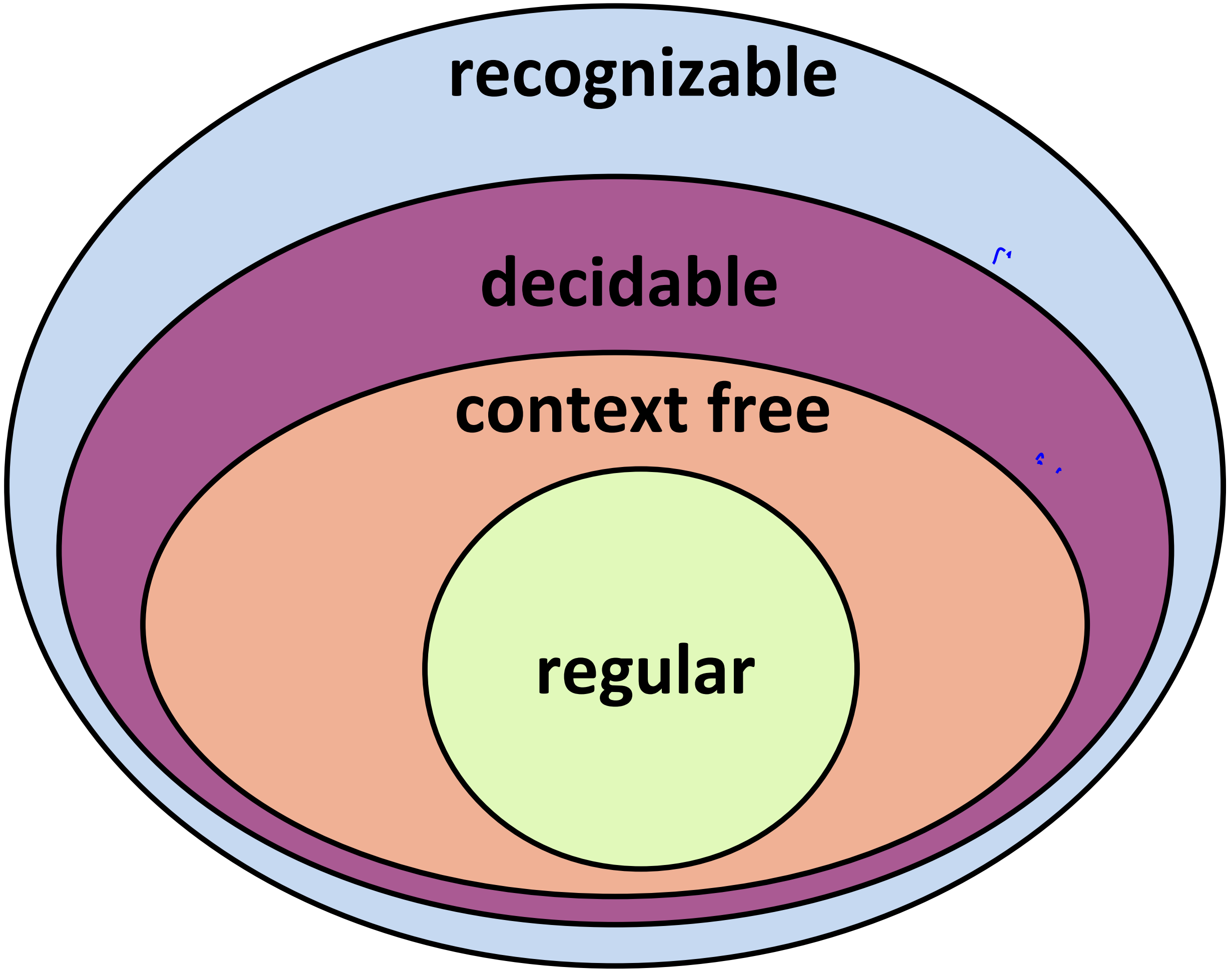
$\boldsymbol{R}$ = decidable languages

$\boldsymbol{RE}$ = Turing-recognizable languages

$\textbf{co}\boldsymbol{RE} = \{\mathrm{L} \mid \overline{L} \text{ is Turing recognizable}\}$

$\overline{A_{TM}}$

Classes of Languages: updated view



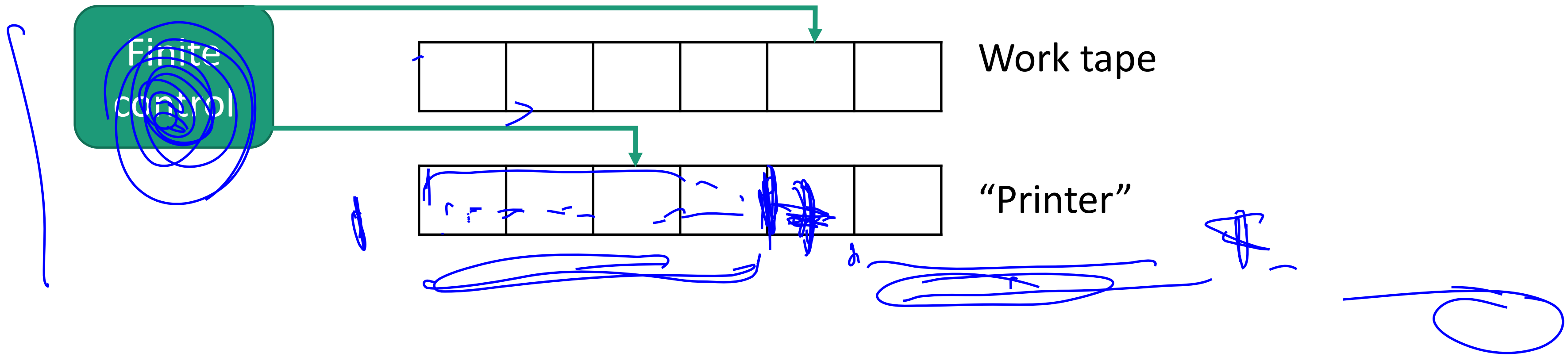
Enumerators

TMs are equivalent to…

- TMs with “stay put”
- TMs with 2-way infinite tapes
- Multi-tape TMs
- Nondeterministic TMs
- Random access TMs
- Enumerators

…

Enumerators



- Starts with two blank tapes
- Prints strings to printer

$L(E) = \{$strings eventually printed by $E\}$

- May never terminate (even if language is finite)
- May print the same string many times

Enumerator Example

1. Initialize $c = 1$
2. Repeat forever:
   - Calculate $s = c^2$ (in binary)
   - Send s to printer
   - Increment c

$L(E) = \{ n^2 \mid n \in \mathbb{N} \}$

What language can ~~an enumerator~~ generate?

this enumerator

Enumerable = Turing-Recognizable

Theorem: A language is Turing-recognizable $\Leftrightarrow$ some enumerator enumerates it

$\Leftarrow$ Start with an enumerator E for A and give a TM

E:

TM (x) =

Run E.

Enumerable = Turing-Recognizable

Theorem: A language is Turing-recognizable $\Leftrightarrow$ some enumerator enumerates it

$\Rightarrow$ Start with a TM M for A and give an enumerator

E: for all $x \in \Sigma^*$

ε

$A(\varepsilon)$ acc : ε

$A(0)$

$A(1)$

$A(00)$

$\vdots$

$\boxed{A_{TM}}$

Reductions

Scientists vs. Engineers

A computer scientist and an engineer are stranded on a desert island. They find two palm trees with one coconut on each. The engineer climbs a tree, picks a coconut and eats.

The computer scientist climbs the second tree, picks a coconut, climbs down, climbs up the first tree and places it there, declaring success.

“Now we’ve reduced the problem to one we’ve already solved.”

Reductions

A reduction from problem A to problem B is an algorithm for problem A which uses an algorithm for problem B as a subroutine

If such a reduction exists, we say "A reduces to B"

Two uses of reductions

Positive uses: If A reduces to B and B is decidable, then A is also decidable

$EQ_{\mathrm{DFA}} = \{\langle D_1, D_2\rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$

Theorem: EQ_{DFA} is decidable

Proof: The following TM decides EQ_{DFA}

On input $\langle D_1, D_2\rangle$, where $\langle D_1, D_2\rangle$ are DFAs:

1. Construct a DFA D that recognizes the symmetric difference $L(D_1) \triangle L(D_2)$
2. Run the decider for E_{DFA} on $\langle D\rangle$ and return its output

Two uses of reductions

Negative uses: If A reduces to B and A is undecidable, then B is also undecidable

$A_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}$

Suppose H decides A_{TM}

Consider the following TM D.

On input $\langle M\rangle$ where M is a TM:

1. Run H on input $\langle M, \langle M\rangle\rangle$
2. If H accepts, accept. If H rejects, reject.

Claim: D decides

$$SA_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM that accepts on input } \langle M\rangle\}$$

Two uses of reductions

Negative uses: If A reduces to B and A is undecidable, then B is also undecidable

Proof template:

1. Suppose to the contrary that B is decidable
2. Using B as a subroutine, construct an algorithm deciding A
3. But A is undecidable. Contradiction!

Halting Problem

$$HALT_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that halts on input } w\}$$

Theorem: $HALT_{\mathrm{TM}}$ is undecidable

Proof: Suppose for contradiction that there exists a decider H for $HALT_{\mathrm{TM}}$. We construct a decider for A_{TM} as follows:

On input $\langle M, w\rangle$:

1. Run H on input $\langle M, w\rangle$
2. If H rejects, reject
3. If H accepts, simulate M on w
4. If M accepts, accept. Otherwise, reject

This is a reduction from A_{TM} to $HALT_{\mathrm{TM}}$

Empty language testing for TMs

$$E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Theorem: E_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for E_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w \rangle$:

1. Run R on input ???

This is a reduction from A_{TM} to E_{TM}

Empty language testing for TMs

$$E_{\mathrm{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

Theorem: E_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for E_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w\rangle$:

1. Construct a TM M' as follows:

2. Run R on input $\langle M'\rangle$

3. If R , accept. Otherwise, reject

This is a reduction from A_{TM} to E_{TM}

Context-free language testing for TMs

$$CFL_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is context} - \text{free}\}$$

Theorem: CFL_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for CFL_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w \rangle$:

1. Construct a TM M' as follows:

2. Run R on input $\langle M' \rangle$
3. If R accepts, accept. Otherwise, reject

This is a reduction from A_{TM} to CFL_{TM}

Context-free language testing for TMs

$$CFL_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is context} - \text{free}\}$$

Theorem: CFL_{TM} is undecidable

Proof: Suppose for contradiction that there exists a decider R for CFL_{TM}. We construct a decider for A_{TM} as follows:

On input $\langle M, w\rangle$:

1. Construct a TM M' as follows:

   M' = "On input x,

   1. If $x \in \{0^n 1^n 2^n \mid n \geq 0\}$, accept
   2. Run TM M on input w
   3. If M accepts, accept."

2. Run R on input $\langle M' \rangle$
3. If R accepts, accept. Otherwise, reject

This is a reduction from A_{TM} to CFL_{TM}