BU CS 332 – Theory of Computation

Lecture 10:

- Turing machines

Reading:

Sipser Ch 3.1,3.2

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  Investigated its "computing power:"
  - Found nice characterizations of the languages recognizable by DFAs
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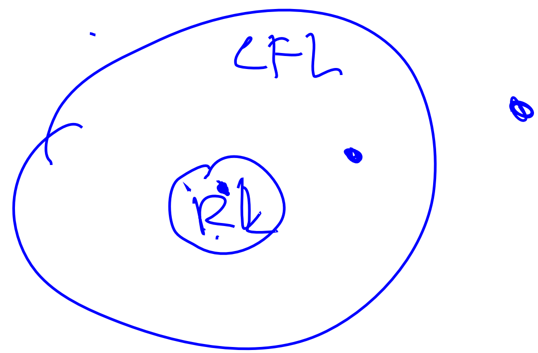
- Modeled computational problems as recognizing languages
- Devised a simple mechanistic "computing device": DFA
  Investigated its "computing power:"
  - Found nice characterizations of the languages recognizable by DFAs
  - Founds languages that are unrecognizable by DFAs (eg $\{0^n1^n | n \geq 0\}$)
  - Inherent limitation: can't count...
- Devised a different "computing device": CFGs
  - Can count and compare, parse math expressions
  - Still limited: has only "short-term memory": Can't recognize $\{0^n1^n0^n \mid n \geq 0\}$

....recap of Act I :

- Starting to see some "hierarchy" of complexity of languages:



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- But: so far, these computing devices look like toy examples...

Whereas we set out to understand the nature of computing...

$0^n 1^n 0^n$

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- But: so far, these computing devices look like toy examples…

Whereas we set out to understand the nature of computing…

?

A Brief History

1900 – Hilbert’s Tenth Problem

Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers.

David Hilbert 1862-1943

1928 – The *Entscheidungsproblem*

The "Decision Problem"

Is there an algorithm which takes as input a formula (in first-order logic) and decides whether it's logically valid?

Wilhelm Ackermann 1896-1962

David Hilbert 1862-1943

1936 – Solution to the *Entscheidungsproblem*

Alonzo Church 1903-1995

"An unsolvable problem of elementary number theory"

Model of computation: λ-calculus (CS 320)

Alan Turing 1912-1954

"On computable numbers, with an application to the *Entscheidungsproblem*"

Model of computation: Turing Machine

Turing Machines

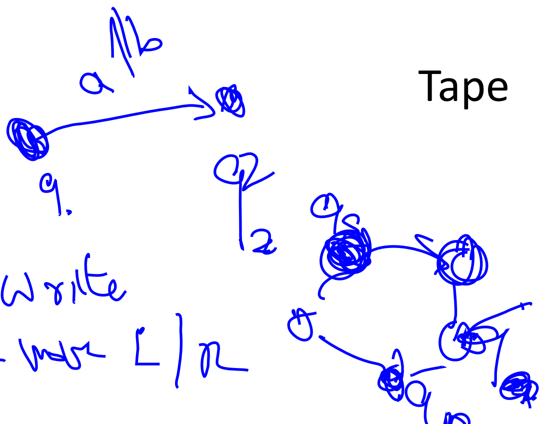
Turing Machines

(In a nutshell: ~~PD~~As with memory…)

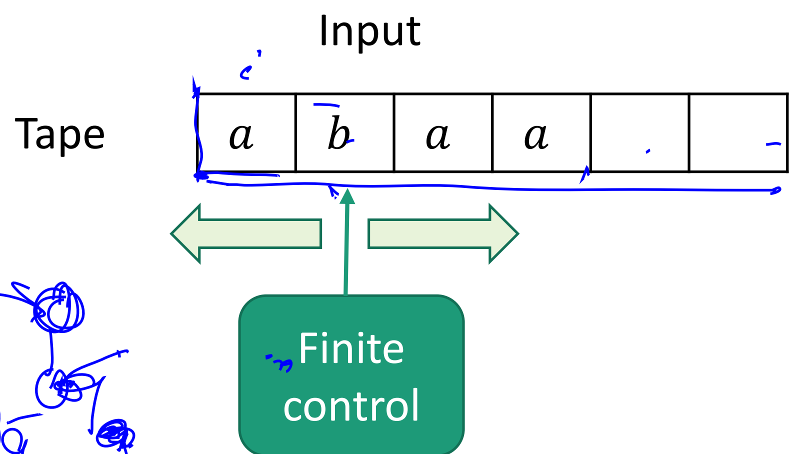
$0^n 1^n$

The Basic Turing Machine (TM)

$\Sigma = \{a, b\}$



– write
– move L | R



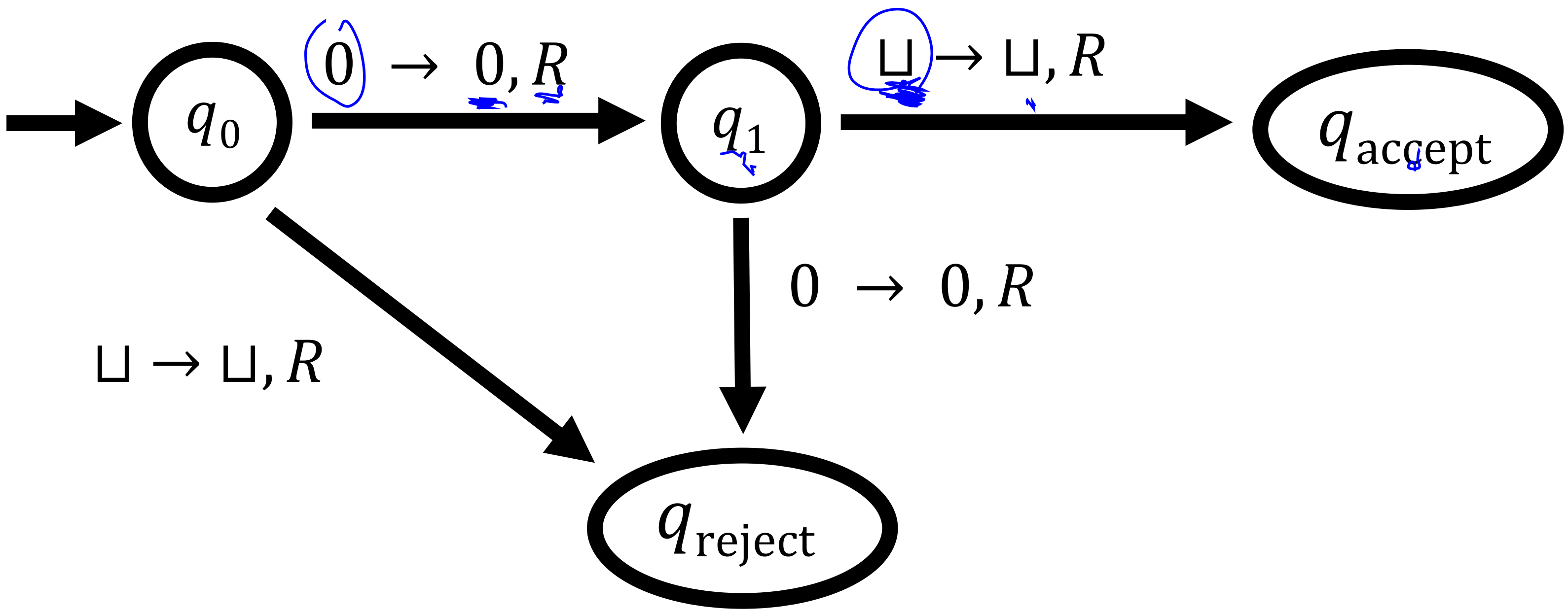
DFA

- Input is written on an infinitely long tape
- Head can both read and write, and move in both directions
- Computation halts when control reaches "accept" or "reject" state

only read
only move right

ends when input spent

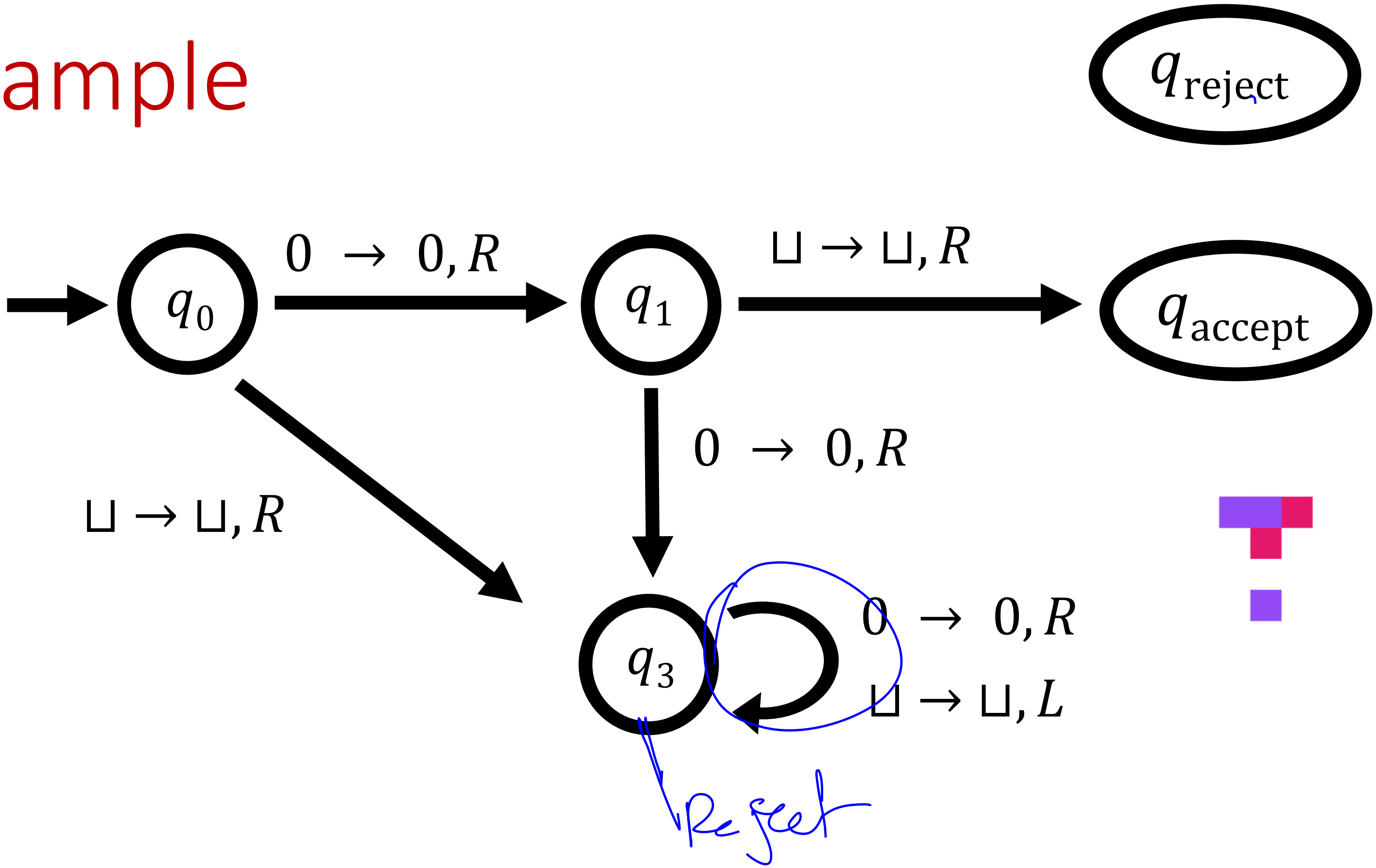
Example



q_0
$0 \to 0, R$
q_1
$\sqcup \to \sqcup, R$
q_{accept}
$0 \to 0, R$
$\sqcup \to \sqcup, R$
q_{reject}

$L = \{0\}$

Example



q_{reject}
q_0
$0 \rightarrow 0, R$
q_1
$\sqcup \rightarrow \sqcup, R$
q_{accept}
$0 \rightarrow 0, R$
$\sqcup \rightarrow \sqcup, R$
q_3
$0 \rightarrow 0, R$
$\sqcup \rightarrow \sqcup, L$
Reject

Three Levels of Abstraction

High-Level Description

An algorithm (like CS 330)

Implementation-Level Description

Describe (in English) the instructions for a TM

- How to move the head
- What to write on the tape

Low-Level Description

State diagram or formal specification

Example

Decide if $w \in A = \{0^{2^n} \mid n \geq 0\}$

High-Level Description

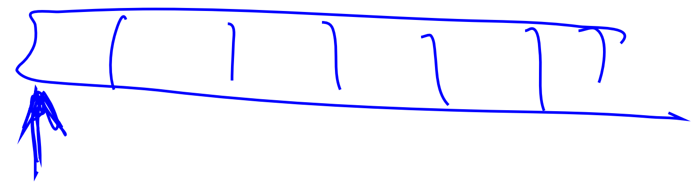
Repeat the following:

- If there is exactly one 0 in w, accept
- If there is an odd number of 0s in w (> 1), reject
- Delete half of the 0s in w

Example

Decide if $w \in A = \{0^{2^n} \mid n \geq 0\}$

Implementation-Level Description

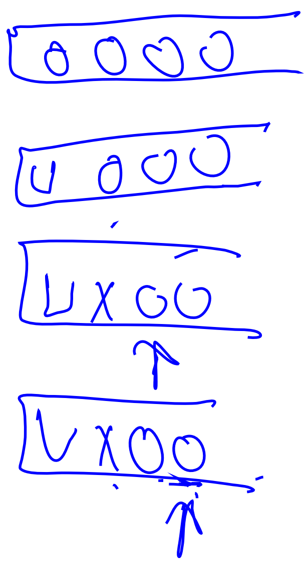


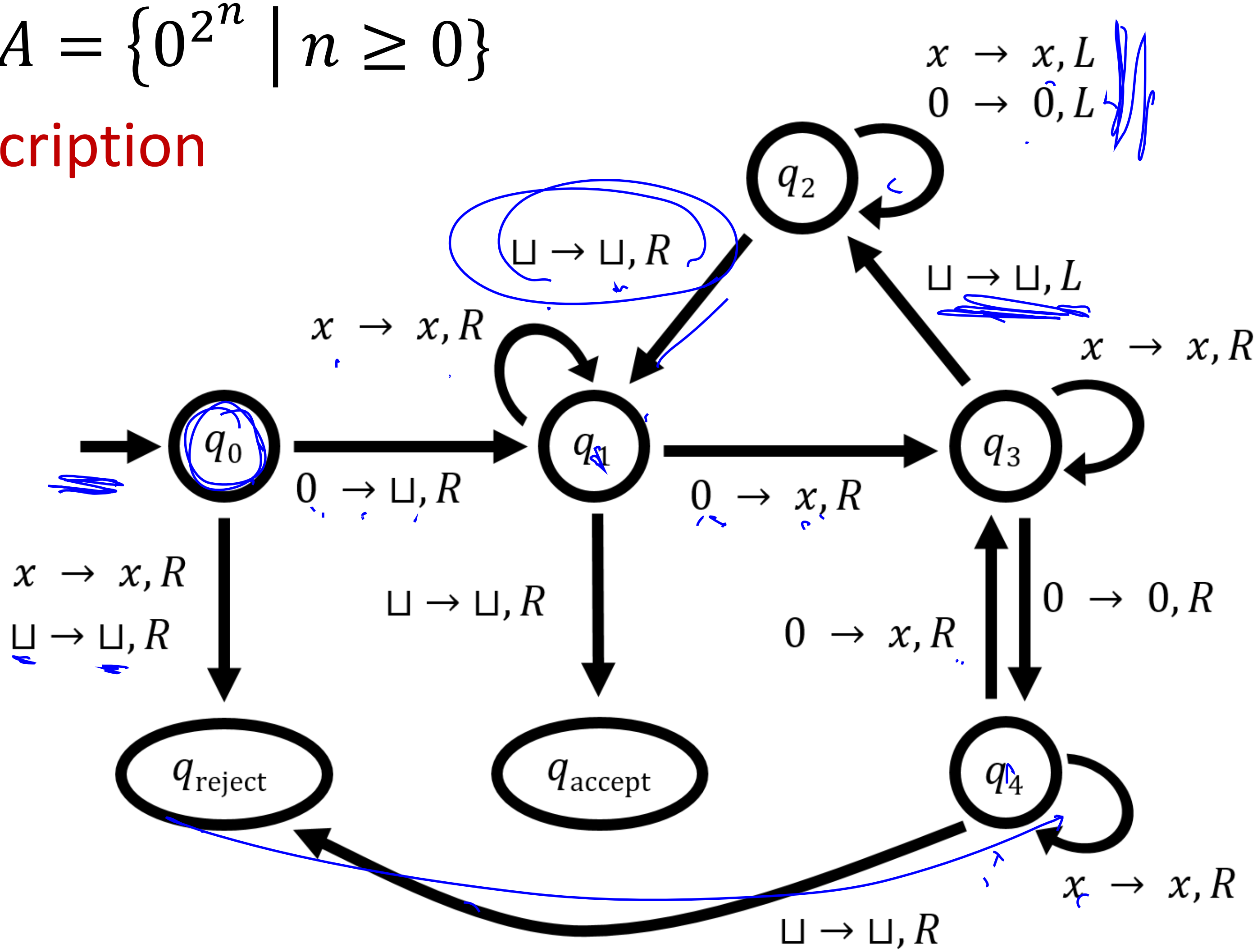
1. While moving the tape head left-to-right:
   a) Cross off every other 0
   b) If there is exactly one 0 when we reach the right end of the tape, accept
   c) If there is an odd number of 0s when we reach the right end of the tape, reject
2. Return the head to the left end of the tape
3. Go back to step 1

Example

Decide if $w \in A = \{0^{2^n} \mid n \geq 0\}$

Low-Level Description





Formal Definition of a TM

A TM is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$

- Q is a finite set of states
- Σ is the input alphabet (does **not** include ⊔)
- Γ is the tape alphabet (contains ⊔ and Σ)
- δ is the transition function

  …more on this later
- $q_0 \in Q$ is the start state
- $q_{\text{accept}} \in Q$ is the accept state
- $q_{\text{reject}} \in Q$ is the reject state ($q_{\text{reject}} \neq q_{\text{accept}}$)

TM Transition Function

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

L means "move left" and R means "move right"

$\delta(p, a) = (q, b, R)$ means:

- Replace a with b in current cell
- Transition from state p to state q
- Move tape head right

$\delta(p, a) = (q, b, L)$ means:

- Replace a with b in current cell
- Transition from state p to state q
- Move tape head left UNLESS we are at left end of tape, in which case don't move