Homework 8

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the course staff if asked. You must also identify your collaborators. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Please practice on exercises and solved problems in Chapter 4 and 5. The material they cover may appear on exams.

Problems There are 4 mandatory problems and one bonus problem.

1. (**DECIDER$_{\mathsf{TM}}$**) Let $\text{DECIDER}_{\mathsf{TM}} = \{\langle M \rangle |\ M$ is a TM that halts on every input$\}$. Prove the following statements.
   (a) $\overline{\text{DECIDER}_{\mathsf{TM}}}$ is not Turing-recognizable (i.e., $\text{DECIDER}_{\mathsf{TM}}$ is not co-Turing-recognizable).
   (b) $\text{DECIDER}_{\mathsf{TM}}$ is not Turing-recognizable.

2. (**Post's correspondence problem**) Tell whether each of the following instances of Post's correspondence problem (PCP) has a solution. Each is presented as two lists x_1 and x_2, and the ith strings on the two lists correspond for each $i = 1, 2, ...$
   (a) $x_1 = (01, 001, 10)$;
   $x_2 = (011, 10, 00)$.
   (in other words, the first piece is $\frac{01}{011}$; the second piece is $\frac{001}{10}$; and the last piece is $\frac{10}{00}$)
   (b) $x_1 = (01, 001, 10)$;
   $x_2 = (011, 01, 00)$.
   (c) $x_1 = (ab, a, bc, c)$;
   $x_2 = (bc, ab, ca, a)$.

3. (**Post's correspondence problem with rotation**) Recall that the Post's correspondence problem (PCP) is where given strings $x_{1,1}, x_{1,2}, ..., x_{1,N}, x_{2,1}, x_{2,2}, ..., x_{2,N}$, find a sequence of indices $(i_1, ..., i_K)$ for some $K \geq 1$ and each $i_K \in [1, N]$ being an integer such that $x_{1,i_1} x_{1,i_2} \cdots x_{1,i_K} = x_{2,i_1} x_{2,i_2} \cdots x_{2,i_K}$.

   Consider the variant RPCP where intuitively you can rotate each domino piece by 180 degrees. For example, the rotation of a piece $\frac{abc}{de}$ would be $\frac{ed}{cba}$. Equivalently, you are given the promise that for any piece $i = 1, ..., N$, you can find its rotated version at some index j, such that $x_{1,i} = x_{2,j}^R$ and $x_{2,i} = x_{1,j}^R$ (the order is reversed since it is a rotation).

   Prove that PCP reduces to RPCP, and conclude that RPCP is undecidable.

   Think but not turn in: Consider another variant FPCP where instead of allowing rotation, we allow flipping, that is the order is not reversed (for example, the flipping of a piece $\frac{abc}{de}$ would be $\frac{de}{abc}$). Prove that FPCP is also undecidable.

4. (**OVERLAP$_{\mathsf{CFG}}$**) Let OVERLAP$_{\mathsf{CFG}}$ $= \{\langle G_1, G_2\rangle \mid G_1$ and G_2 are CFGs and $L(G_1) \cap L(G_2) \neq \emptyset\}$. Show that OVERLAP$_{\mathsf{CFG}}$ is undecidable by giving a reduction from the Post's Correspondence Problem.

   Hint: Given an instance
$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\}$$
   of the Post's Correspondence Problem, construct CFGs G_1 and G_2 with the rules
$$\begin{aligned} G_1 &: T \to t_1 T \sigma_1 \mid \cdots \mid t_k T \sigma_k \mid t_1 \sigma_1 \mid \cdots \mid t_k \sigma_k \\ G_2 &: B \to b_1 B \sigma_1 \mid \cdots \mid b_k B \sigma_k \mid b_1 \sigma_1 \mid \cdots \mid b_k \sigma_k \end{aligned}$$
   where $\sigma_1, ..., \sigma_k$ are new alphabet symbols, and T, B are the starting nonterminal in their corresponding CFG. Prove that this reduction works.

5. (**Bonus, bounded Turing machines**) A *bounded Turing machine* (BTM) is a Turing machine that cannot move its tape head beyond the end of the input. In particular, on given an n-symbol string, the input tape will be initialized to having this string and in addition the blank symbol at the very end to denote the end of input; if the tape head is on $(n+1)$-st cell and it attempts to move right, the tape head will stay at the same location.

   (a) Show that BTMs can decide any context-free language.
   Hint: Prove that any k-tape BTM can be simulated by a single-tape BTM with a larger tape alphabet. Use Chomsky normal form.

   (b) Show that BTMs are strictly more powerful than CFGs. Namely, give an example of a language that can be recognized by a BTM, but not generated by a CFG.
   Give a high-level description in English of a BTM recognizing your language.

   (c) Prove that $\mathrm{A}_{BTM} = \{\langle M, w\rangle | M$ is a BTM that accepts on input $w\}$ is decidable, and conclude that TMs are strictly more powerful than BTMs.

   (d) Prove that $\mathrm{E}_{BTM} = \{\langle M\rangle | M$ is a BTM that recognizes the empty language$\}$ is undecidable. (recall that E_{CFG} is decidable!)
   Hint: Think how the method of computation histories could help with the reduction.