CS 332: *Theory of Computation* — Prof. Ran Canetti
Boston University — October 27, 2020

Homework 7 – Due Tuesday, November 3rd, 2020 before 2:00PM

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the course staff if asked. You must also identify your collaborators. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Please practice on exercises and solved problems in Chapter 3. The material they cover may appear on exams.

Problems There are 3 mandatory problems and one bonus problem.

1. (**Countable and uncountable sets**) Prove or disprove that each of the following sets is countable.

   (a) (**5 points**) The integer set $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$.

   (b) (**5 points**) The set of degree-d integer-coefficient univariate polynomials $\mathcal{P}_d$, that is each element in the set is a polynomial in variable x of the form $c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_dx^d$, where d is a non-negative integer and $c_0, ..., c_d$ are integers, called coefficients.

   (c) (**5 points**) The set of all integer-coefficient univariate polynomials $\mathcal{P} = \bigcup_{d \geq 0} \mathcal{P}_d$.

   (d) (**5 points**) The set of all finite sets of real numbers.

   (e) (**5 points**) The set of all possible English sentences.

   (f) (**5 points**) The set of all finite languages over the binary alphabet $\{0, 1\}$.

   (g) (**10 points**) The set of all languages over the unary alphabet $\{0\}$.

2. (**20 points**) A TM *correctly sorts* if, given a comma-separated list of binary numbers, it halts with the sorted (from smallest to largest) version of the list on its tape. (It does not matter what it does on other inputs.) The goal of this question is to show that it is impossible to have an algorithm that correctly determines whether a given TM *correctly sorts*.

   (a) (**6 points**) Formulate this problem as a language L_{sort}.

   (b) (**10 points**) Describe a reduction from $HALT_{TM}$ to L_{sort}. That is, describe a computable translation $T : \{0, 1\}^* \to \{0, 1\}^*$ from descriptions of standard TMs to descriptions of purported sorting machines.

   (c) (**2 points**) Show that if M halts on the empty input then $T(\langle M \rangle)$ is a description of a sorting machine.

   (d) (**2 points**) Show that if M does not halt on the empty input then $T(\langle M \rangle)$ is not a description of a sorting machine.

3. (**Decidable/undecidable languages**) For each of the parts, formulate the given problem as a language and either prove or disprove it is decidable.
   Hint: Exactly one of them is decidable and the other is not.

(a) (**20 points**) You are given a TM and you would like to determine whether there exists some input w on which this TM moves its head to the *left* from the tape cell 2020. (We number the tape cells from left to right, starting from 1.) Note that w is not given to you.

(b) (**20 points**) You are given a TM and you would like to determine whether there exists some input w on which this TM moves its head to the *right* from the tape cell 2020.

4* (**Optional, no collaboration is allowed**) In this problem, you are asked to think about *LOSS* operations on languages. Each LOSS operation is specified by a set Σ of symbols. When the "*LOSS of* Σ" operation, denoted by $LOSS_\Sigma$, is applied to a string w, all characters in Σ disappear from w. For example, $LOSS_{\{\mathtt{1,3}\}}(\mathtt{121023}) = \mathtt{202}$ and $LOSS_{\{\mathtt{1,3}\}}(\mathtt{241222}) = \mathtt{24222}$, whereas $LOSS_{\{\mathtt{1,3}\}}(\mathtt{24222}) = \mathtt{24222}$. To apply $LOSS_\Sigma$ to a language, we apply it to every string in the language. For example, $LOSS_{\{\mathtt{0,1,3}\}}(\mathtt{0}^*\mathtt{1}^*\mathtt{2}^*\mathtt{3}) = \mathtt{2}^*$. More formally,

$$LOSS_\Sigma(L) = \{LOSS_\Sigma(w) \mid w \in L\}.$$

(a) Prove that the class of regular languages is closed under the LOSS operations.

(b) Prove that the class of decidable languages is not closed under the LOSS operations.