Homework 6 – Due Tuesday, October 27th, 2020 before 2:00PM

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the course staff if asked. You must also identify your collaborators. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Please practice on exercises and solved problems in Chapter 3. The material they cover may appear on exams.

Problems There are 4 mandatory problems and one bonus problem.

1. (**20 points**) Recall that languages $L, L' \subseteq \Sigma^*$ are complementary if $L' = \Sigma^* - L$. Consider the problem of determining if the languages of two given DFAs are complementary. Formulate this problem as a language $\text{COMPLEMENT}_{\mathsf{DFA}}$ and show that it is decidable.

2. (**20 points**) Consider the following *3D cell Turing machine*, which is the natural generalization of a one-dimensional Turing machine to the three-dimensional physical space that we live in. 3D cell Turing machine works on an infinite 3D space of cells, with each cell holding an element from the cell alphabet Γ; and the transition function is $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, U, D, F, B\}$, where L, R, U, D, F, B means moving left/right/up/down/forward/backward (by one cell) respectively. (You can also think of this model as a bug flying in a space filled with hard drives, where the bug is reading from and writing to the drives as it flies around.) Show that the set of languages that are recognizable by a 3D cell Turing machine is **RE**, namely the set of Turing-recognizable languages.

3. (**Closure properties**)

   (a) (**10 points**) Show that the class of decidable languages is closed under concatenation.

   (b) (**10 points**) Show that the class of Turing-recognizable languages is closed under star

   (c) (**10 points**)
   Given a string w over $\Sigma = \{0, 1\}$, define

   $$\text{BALANCE}(w) = \begin{cases} (01)^x & \text{if } x = y, \\ (01)^y 0^{x-y} & \text{if } x > y, \\ (01)^x 1^{y-x} & \text{if } x < y, \end{cases}$$

   where x is the number of 0s in w and y is the number of 1s in w.
   For example, BALANCE(000111)=010101 and BALANCE(0100000111)=0101010100. Note that w and BALANCE(w) have the same number of 0s and 1s, but in the beginning of the string BALANCE(w), the 0 and 1 characters alternate until one of them runs out.
   Define the corresponding operation on languages:

   $$\text{BALANCE}(L) = \{\text{BALANCE}(w) \mid w \in L\}.$$

   Show that the class of decidable languages is closed under Balance.

(d) (**10 points**) Show that the class of Turing-recognizable languages is closed under BALANCE.

For decision, think about complement, union (solution to 3.15(a) on p. 191 in Sipser), intersection and star on your own. For recognition, think about union (solution 3.16(a) on p. 191 in Sipser), intersection and concatenation on your own.

4. (**20 points**) Let M be a TM. Show that there exists a TM M' with three states and two tapes such that $L(M) = L(M')$. Furthermore, M' is a decider if and only if M is a decider.

5* (**Bonus, no collaboration**) Let A be a Turing-recognizable language which is not decidable. (We will prove later in the course that such languages exist.) Consider a TM M that recognizes A. Prove that there are infinitely many strings on which M does not halt.